MATHEMATICS

"ON MATRIX TRANSFORMATIONS OF CERTAIN SEQUENCE SPACES" – CORRECTION AND NOTE

BY

H. R. CHILLINGWORTH

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In Theorem IV of the paper to which we refer, published in Proceedings, Series A, **60**, no. 5 and Indag. Math. **19**, no. 5, 1957, the assumption, five lines from the foot of the page, "that p is not in a W-set for β" is not valid for certain types of convergence free space, namely those in which the distribution, and not the position, of certain zero terms is the characteristic feature.

In order to correct the theorem, we classify convergence-free spaces as follows. (For definition of the particular spaces to which we refer below, see [1], Chapter 10.)

Class I. Spaces in which every sequence contains zero elements corresponding to a certain set of suffixes (fixed for the space), other elements remaining arbitrary. (Examples: 0_1, 0_2)

Class II. Spaces in which every sequence is of form $x+y$, where x is a sequence in ϕ, and y a sequence in a (fixed) space of Class I. (ϕ, 0_1, 0_2)

Class III. Spaces in which the distribution and not the position of zero terms is specified. (δ)

Class IV. Spaces which combine the above characteristics. (For example, a space in which sequence elements with even suffixes are zero, and zero elements with odd suffixes are distributed as in δ.)

Class V. The space σ, in which no stipulation with regard to zero elements is made.

Theorem IV of the paper, as given, applied only to spaces of Class I, and thus the proof of Theorem V was valid only for spaces of this class.

However, since spaces of classes II and V are all easily shown to be perfect, Theorem V is (trivially) true for these classes also. But since $\sum(\delta)$ contains all the diagonal matrices, while $\delta^{**} \rightarrow \delta$ ($\equiv \sigma \rightarrow \delta$) clearly does not contain these matrices, the theorem breaks down in this case.

Nevertheless, by means of the following lemma (the proof of which is due to J. B. TATCHELL) we show that the theorem does in fact apply to a large class of spaces α and all convergence free spaces.

Lemma. *If* $A=(a_{m,k})$ *is a matrix with rows in* σ_1, *and containing no zero rows, then there exists a sequence* $\{u_k\}$ *in* σ_∞ *with the property that* $\sum_{k=1}^{\infty} a_{mk}u_k \neq 0$, $m=1, 2, \ldots$.

Proof. We may choose a bounded sequence $\{u_k^{(1)}\}$ and a positive number λ_1 such that $|\sum_{k=1}^{\infty} a_{1k}u_k^{(1)}|>\lambda_1$. We may then find a second sequence $\{u_k^{(2)}\}$, and a suitable positive number λ_2, such that $|\sum_{k=1}^{\infty} a_{1k}u_k^{(2)}|>\lambda_1$, $|\sum_{k=1}^{\infty} a_{2k}u_k^{(2)}|>\lambda_2$, while $|u_k^{(1)}-u_k^{(2)}|\leqslant\frac{1}{2}$ for each k.

Proceeding in this way, we may define by induction a set of sequences $\{u_k^{(n)}\}$, $n\geqslant 1$, and a sequence $\{\lambda_n\}$ of positive numbers, such that $|\sum_{k=1}^{\infty} a_{mk}u_k^{(n)}|>\lambda_m$ whenever $1\leqslant m\leqslant n$, and $|u_k^{(n)}-u_k^{(n+1)}|\leqslant\frac{1}{2^n}$ for each k, $(n=1, 2, \ldots)$.

Now, for each fixed k,

$$(1)\qquad \left\{\begin{aligned} |u_k^{(p)}-u_k^{(q+1)}| &= |\sum_{\nu=p}^{q}(u_k^{(\nu)}-u_k^{(\nu+1)})| \\ &\leqslant \sum_{\nu=p}^{\infty}\frac{1}{2^\nu} \\ &= \frac{1}{2^{p-1}}. \end{aligned}\right.$$

Hence $\lim_{n\to\infty} u_k^{(n)}=u_k$ is defined for each k.

It follows from (1) that (a) each sequence $\{u_k^{(n)}\}$ is bounded, as is seen by letting $p=1$;

(b) $\{u_k\}$ is bounded, and

$$(2)\qquad |u_k-u_k^{(n)}| = \lim_{q\to\infty}|u_k^{(q)}-u_k^{(n)}| \leqslant \frac{1}{2^{n-1}} \text{ for each } k.$$

Since rows of A are in σ_1, $\sum_{k=1}^{\infty} a_{mk}u_k$ exists for every m, and, using (2), it is readily shown that

$$|\sum_{k=1}^{\infty} a_{m,k}u_k| = \lim_{n\to\infty}|\sum_{k=1}^{\infty} a_{m,k}u_k^{(n)}|\geqslant\lambda_m \qquad (m=1,2,\ldots).$$

Thus $\sum_{k=1}^{\infty} a_{mk}u_k\neq 0$ for any m. This proves the lemma.

Theorem. *If* α *is a sequence space which is normal and contains a sequence without zero elements, and* β *is convergence-free, a matrix* $A=(a_{n,k})$ *is in* $\alpha\to\beta$ *if, and only if,* (i) *the row suffixes of the non-zero rows of* A *form a W-set for* β. (ii) rows of A are in α^*.

Proof. Let $\{x_k\}$ be a sequence in α which contains no zero elements. Let $\{v_k\}$ be any sequence in σ_∞. Then $\{x_kv_k\}$ is a sequence in α, which is normal. Hence if A belongs to the matrix space $\alpha \to \beta$, $\{y_n\} \equiv \{\sum_{k=1}^{\infty} a_{nk}x_k v_k\}$ is a sequence in β. But if the non-zero rows of A have suffixes which do not form a W-set for β it follows that $\{\sum_{k=1}^{\infty} a_{nk}x_k v_k\}=0$ for some values of n which do not correspond to zero rows.

Let the suffixes of non-zero rows of A be $n_1, n_2, \ldots$ and replace $a_{n_pk}x_k$ by b_{n_pk} $(p, k=1, 2, \ldots)$. Then $\{\sum b_{n_pk}v_k\}=0$ for some values of p, where $\{v_k\}$ is an arbitrary sequence in σ_∞, and rows of $B=(b_{n_p,k})$ are in σ_1, since rows of A are in α^*.

It now easily follows from the lemma that this condition is not satisfied for every $\{v_k\}$ in σ_∞; thus a sequence exists in α such that its A-transform is not in β. Hence the given condition (i) is necessary; and, together with (ii), it is obviously sufficient. This proves the theorem, since the necessity of condition (ii) follows from the definitions.

Corollary. *If* (1) $\alpha \leqslant \lambda \leqslant \alpha^{**}$, *and* α *is a normal sequence space containing a sequence without zero elements, and if* (2) β *is convergence-free, and* (3) $M(\lambda, \beta)$ *is the matrix space which transforms sequences in* λ *to sequences in* β, *then* $\alpha \to \beta = \lambda \to \beta = M(\lambda, \beta) = \alpha^{**} \to \beta$.

This corollary is proved along the lines of theorem V of the paper quoted, which it replaces.

Birkbeck College,
London, W.C. 1

REFERENCE

[1]. Cooke, R. G. *Infinite matrices and sequence spaces.* (Macmillan, 1950).